J. Fluid Mech. (1985), *vol.* 150, *pp.* 357–380
Printed in Great Britain

Observations of rapidly flowing granular-fluid materials

By DANIEL M. HANES† AND DOUGLAS L. INMAN
Center for Coastal Studies, University of California, San Diego, La Jolla, California 92093

(Received 27 September 1983 and in revised form 6 August 1984)

The rapid shearing of a mixture of cohesionless glass spheres and air or water was studied in an annular, parallel-plate shear cell designed after Savage (1978). Two types of flow were observed. In the first type of flow the entire mass of the granular material was mobilized. At high shear rates the shear and normal stresses were found to be quadratically dependent upon the mean shear rate (at constant volume concentration), in general agreement with the observations of Bagnold (1954) and Savage & Sayed (1984), and the 'kinetic' theory of Jenkins & Savage (1983). The stresses were found to be weakly dependent on the volume concentration up to approximately 0.5, and strongly dependent above this concentration. For flows in which water was the interstitial fluid, the ratio of the shear stress to the normal stress was slightly higher (than in air), and the stresses at lower shear rates were found to be more nearly linearly related to the shear rate. It is suggested that these effects are contributed to by the viscous dampening of grain motions by the water. The second type of flow was distinguished by the existence of an internal boundary above which the granular material deformed rapidly, but below which the granular material remained rigidly locked in place. The thickness of the shearing layer was measured to be between 5 and 15 grain diameters. The stress ratio at the bottom of the shearing layer was found to be nearly constant, suggesting the internal boundary is a consequence of the immersed weight of the shearing grains, and may be described by a Coulomb yield criterion. A scaled concentration is proposed to compare similar data obtained using different-sized materials or different apparatus. An intercomparison of the two types of flow studied, along with a comparison between the present experiments and those of Bagnold (1954) and Savage & Sayed (1984), suggests that the nature of the boundaries can have a significant effect upon the dynamics of the entire flow.

1. Introduction

Diverse phenomena such as rockslides, the bedload transport of coarse-grained sediments, and the industrial handling of seeds, slurries and pharmaceutical powders provide ample motivation to develop theories which describe the bulk flow of particles at high shear rates. Theoretical studies on the constitutive behaviour of rapidly flowing granular materials have increased in number and sophistication over the last few years (e.g. Lun *et al.* 1984; Jenkins & Satake 1983). Experimental observations of the details of such flows are unfortunately quite sparse because of the difficulty in obtaining reliable measurements of particle velocities and concentrations within

† Present address: Division of Applied Marine Physics, Rosenstiel School of Marine and Atmospheric Sciences, Miami, Florida 33149.

the flow. The lack of reliable measurements has made it difficult to confirm or refute the various hypotheses proposed. However, viscometric-type measurements reported by Bagnold (1954), Savage & McKeown (1983) and Savage & Sayed (1984), and the numerical simulations of Campbell & Brennen (1982) have provided much insight into the mechanics of rapid granular flows. Their observations have suggested that granular collisions comprise the primary mechanism for momentum transfer, resulting in both a shear stress and a normal stress which vary as the square of the mean shear rate (at a constant volume concentration). Experiments have proven difficult to duplicate however, even in similar apparatus.

The present work describes observations of granular-fluid materials flowing rapidly in an annular trough between parallel rotating plates. The experiments generally involve flows in which the effects of the interstitial fluid (air) are small, but some observations of flows in which the interstitial fluid (water) was non-negligible were also made. For fully shearing flows the dependence of the stresses upon both the shear rate and the volume concentration are described. These observations overlap and extend the observations of Savage & Sayed (1984). Observations are also made of flows in which a shearing zone exists above a nonshearing 'plug' zone. This appears to be a natural consequence of the self-weight of the grains in a gravity field.

2. Experimental considerations

2.1. *Apparatus*

An apparatus capable of measuring the large stresses and dilatancy effects typical of granular-fluid materials undergoing rapid shear deformation was constructed after the design of Savage (1978). Material is sheared in an annular trough between parallel, horizontal, rotating plates. The primary modification from Savage's design is that the outer wall is clear, allowing for direct visual observations. The shear cell allows for measurements of the shear and normal stresses developed by shearing granular-fluid materials at a measurable volume concentration and shear rate.

The shear cell is shown schematically in figure 1. Shearing takes place in a 4.4 cm wide annular gap centred at a radius of 12.4 cm. The area of a horizontal slice through the annulus was 341 cm^2. The at-rest volume of the material studied ranged from about 200 to 1000 cm^3.

The upper and lower shearing surfaces were roughened by cementing with epoxy one to two grain layers of the material being sheared to each surface. The sidewalls were smooth aluminium and smooth polycarbonate, both coated with a thin layer of Teflon. The side walls rotated rigidly with the lower plate, which was belt-driven by a variable-speed motor. The upper plate was free to move vertically, but restrained from rotating by a linkage to a strain gauge. A counterweight system allowed variation of the applied normal stress.

The outer wall was clear acrylic, protected by a thin, replaceable, clear polycarbonate inner lining. This allowed for direct visual observation of the boundary between the shearing and non-shearing regions, when such a boundary existed.

2.2. *Material description*

The granular materials used in this study were spherical glass beads and natural sand. The sand was obtained from Boomer Beach in La Jolla, California. The sand is well-rounded quartz and is described by Inman (1953). The sand was washed and sieved into the size fraction that was used in the experiments. The upper and lower sieve mesh sizes were 0.594 and 0.500 mm, giving the sand an average diameter of 0.55 mm

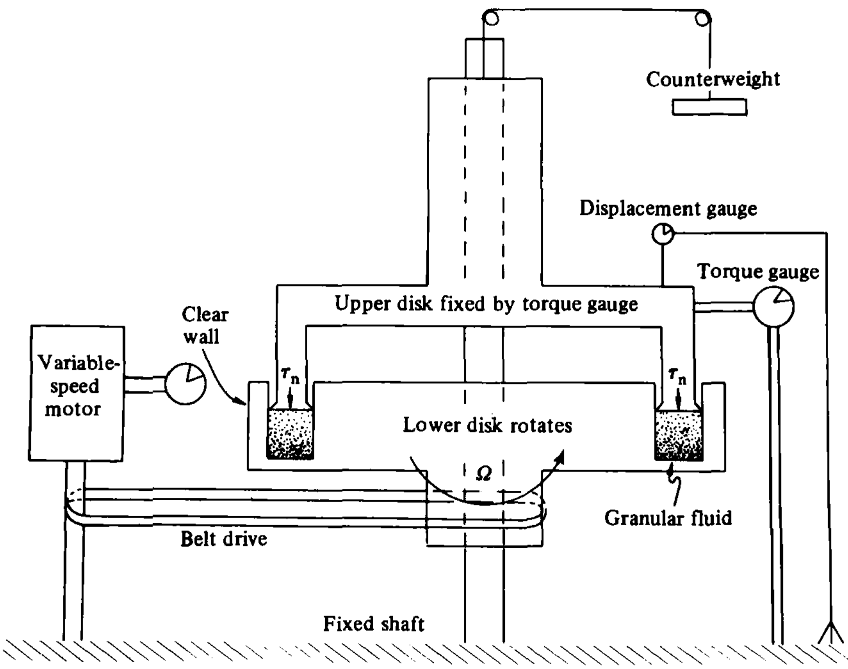

FIGURE 1. Schematic of annular shear cell.

Material	Diameter (mm)	Density (g cm^{-3})
Spherical glass beads	$1.0 \leqslant D < 1.19$	2.48
	$1.7 \leqslant D < 2.0$	2.78
Natural well-rounded quartz sand	$0.50 \leqslant D < 0.59$	2.65

TABLE 1. Characteristics of the granular materials

for its arithmetic class mark. The glass beads were purchased from Potters Industries of New Jersey and the Ferro Corporation of North Carolina. Because the beads initially had a large variation in the degree of their sphericity, only the most spherical grains were used. These grains were separated from the less-spherical ones by rolling them down a smooth plane inclined at approximately two degrees from horizontal. The beads were also sieved to obtain nearly uniform sizes. The upper and lower mesh sizes for the average 1.1 mm spheres were 1.19 mm and 1.0 mm. The upper and lower mesh sizes of the average 1.85 mm spheres were 2.0 mm and 1.7 mm. The densities of the various materials were determined by measuring the amount of water displaced in a 100 cm^3 graduated cylinder by a known mass of material. The physical characteristics of the materials are given in table 1.

2.3. *Experimental plan*

The majority of the experiments involved the shearing of spherical glass beads in air. These experiments can be used to evaluate theories describing the rapid flow of nearly elastic spherical particles in which the effects of the interstitial fluid are negligible. Because many granular flows of interest involve non-spherical grains, and immersing fluids with densities comparable to that of the grains, series of experiments were also run using water as the immersing fluid, and sand as the granular material. Depending upon the amount of material used and the conditions to which it was subjected, two types of flow were observed. In one case, all of the material in the annulus sheared. In the second case, only a portion of the material sheared, while the rest of the material remained interlocked in rigid-body rotation below the shearing grains. These two cases are referred to as fully and partially shearing.

The basic procedures for the experiments were as follows.

A preweighed amount of granular material was placed in the shear cell. For the partially shearing experiments, chalk dust was placed on the sidewall. After applying a known normal stress, the material was prestressed and consolidated by slowly shearing it back and forth for five cycles. The initial at-rest volume of the material was recorded along with the zero offset of the strain gauge. The shearing was begun by slowly increasing the speed of rotation of the bottom assembly to the desired level. After steady state was reached (about 5–15 s), the period of rotation, the output of the strain gauge, and the displacement of the upper plate were recorded. The motor speed was then reduced slowly to zero. The level of no shear was then recorded for the partially shearing experiments. The material was then poststressed and consolidated as before. The final at-rest volume of the granular material and the zero-offset of the strain gauge were recorded again. If these values deviated from the initial values, then the averages of the two were used in later calculations.

The entire process was repeated for a variety of applied normal stresses and speeds. For the fully shearing experiments the speed was adjusted to maintain the same volume concentration over a range of applied normal stresses. For these experiments, the normal stress and the associated speeds of rotation were changed without bringing the system to a halt, after it was determined that the results thus obtained were the same as when the shearing was stopped after each run.

For the experiments in which water was the interstitial fluid, water was added to the system as the material dilated. This was done in order to maintain the fluid level even with the surface of the upper plate, thus avoiding any buoyancy effect on the plate.

2.4. *Measurement systems*

The variables that were directly measured were the mass and the initial (at-rest) volume of the granular material, the change in volume of this material during shear, the force exerted by the upper plate upon the strain gauge, the rotational period of the lower plate and attached sidewalls, and the thickness of the shearing region while shear was occurring and afterwards, when the material was at rest.

The change in volume during shearing was calculated by measuring the vertical displacement of the upper plate with a machinist's gauge. Note that this measurement was made without stopping the rotation. The torque exerted on the upper plate was measured by means of a strain gauge mounted on a steel beam. This instrument, with its associated electronics, produces a signal linearly proportional to the transverse force applied to the beam. The torque exerted on the upper plate is given by

$$\text{torque} = \int_{r_1}^{r_2} T_{xz}(r)\, 2\pi r\, r\, \mathrm{d}r, \tag{1}$$

where $T_{xz}(r)$ is the shear stress exerted by the granular material upon the upper plate at a radial distance r. In performing this calculation it was assumed that $T_{xz}(r)$ was constant across the annular gap for a given experiment. If, in fact, $T_{xz}(r)$ was a linear

Measurement	Resolution	Accuracy (%)
Mass of granular material	0.1 g	0.001
Displacement of upper plate	0.0013 cm	1.7
Location of internal boundary	0.05 cm	5
Shear stress	10 dyn cm^{-2}	0.2
Normal stress	70 dyn cm^{-2}	1.4
Nominal shear rate	0.5 s^{-1}	1.4
Nominal volume concentration	0.017	3.8

TABLE 2. Maximum errors in the measurement systems

or quadratic function of r, the assumption of a constant T_{xz} results in errors of 2 and 4 % respectively for T_{xz} (calculated from (1)) at the mean radius.

The speed of rotation of the lower plate, and hence the maximum velocity, was measured by an electronic timer coupled to a magnetically actuated switch. A magnet was attached to the rotating plate. Each time the magnet passed the switch, the timer triggered and measured the elapsed time until the next trigger.

The thickness of the shearing region was measured by applying chalk dust to a section on the inside of the outer wall. The shearing grains wiped the chalk dust off the wall, while the non-shearing grains left the chalk dust unaltered. This method is believed to give unbiased measurements of the level of no shear, because the interaction between a grain and the sidewall is relatively slippery compared with the interaction between the grain and a neighbouring grain within the flow. In some preliminary experiments, chalk dust was applied to both the inner and the outer walls. The thicknesses thus obtained generally agreed with one another to within one grain diameter. The thickness of the shearing region, denoted by Z_0, was used to calculate the nominal shear rate and volume concentration for the partially shearing experiments.

The resolution of each of these measurement systems are summarized in table 2. The accuracies are also expressed as a fraction of a typical measurement.

2.5. *Approximations and assumptions*

The primary areas of concern are the effects of rotation, the effects of the interstitial fluid, unsteadiness in the flow, and the influences of a finite-sized apparatus.

The experimental apparatus, because of its rotational nature, introduces the possible complications of radial dependence of the field variables. The effects of radial acceleration were always present to some extent, but rotational speeds and applied normal stresses were selected to insure that the centrifugal stress was always much less than the normal stress. Savage & Sayed (1984) measured secondary (radial) flows of granular-fluid materials in a nearly identical apparatus. They concluded these flows were quite small compared with the primary flow, and did not affect the stress measurements. Since the interstitial fluid does not support an anisotropic normal stress, the centrifugal forces could have a significant effect upon the dynamics of the interstitial fluid.

The relative importance of the interstitial fluid on the dynamics of the grains can be estimated by applying the experimental results of Bagnold (1954). Bagnold found that for granular concentrations greater than approximately 8 % the shear resistance of a granular fluid in motion was always much greater than the viscosity of the purely

	Bagnold numbers
In air	114000–161000
In water	
1.1 mm spheres	
fully shearing	348–2500
partially shearing	273–882
1.85 mm spheres	
fully shearing	870–3422
partially shearing	247–1186
0.55 mm sand	
partially shearing	280–666

TABLE 3. Range in Bagnold numbers for the various sets of experiments, where $B = \rho_s \mu^{-1} \lambda^{\frac{1}{2}} D^2 \,dU/dz$. For $B > 450$ the data lie in the grain-inertia region.

fluid phase. Bagnold separated the behaviour of a shearing granular fluid into two limiting cases, the macroviscous regime and the grain-inertia regime.

In the macroviscous regime stresses are transmitted by interstitial fluid friction, and are therefore dependent upon the fluid viscosity, but independent of the grain density. In the grain-inertia regime stresses are transmitted by intergranular collisions, and are therefore independent of the fluid viscosity but dependent upon the grain density. The Bagnold number, defined by (2) and representing the ratio of the inertial to the viscous forces, can be used to characterize the flow as either macroviscous, grain-inertial or transitional:

$$B = \rho_s \mu^{-1} \lambda^{\frac{1}{2}} D^2 \frac{dU}{dz}, \tag{2}$$

where dU/dz is the mean shear rate, ρ_s is the density of the granular material, D is the grain diameter, μ is the interstitial fluid viscosity, and λ, the linear concentration, is the ratio of the grain diameter to the mean free separation distance between grains. λ is given as a function of the volume concentration by

$$\lambda = [(N^*/N)^{\frac{1}{3}} - 1]^{-1}, \tag{3a}$$

where N^* is the maximum possible concentration of the granular material. Bagnold found that for $B < 40$ the flow is in the macroviscous regime, and for $B > 450$ it is in the grain-inertia regime. Bagnold refers to the region between these limits as the transition region. Both the granular materials and the geometry of Bagnold's experiments differed from the present work, so the exact values of the limiting Bagnold numbers for the two regimes may be slightly different. In the present experiments, granular concentrations are always greater than 0.2. Bagnold numbers can be computed based upon mean granular concentration and nominal shear rates. These numbers are given in table 3. Clearly all the flows involving air as the interstitial fluid are in the grain-inertia regime, while flows in which water is the interstitial fluid range from the transitional to the grain-inertia regime.

In general, the flows reached a steady rate (fluctuations within approximately $\pm 5\%$ of the mean) after a few seconds of shearing. However, there were two identifiable sources of unsteadiness. The first resulted from an imperfectly designed apparatus; the second is believed to be a consequence of the mechanics of deformation of some granular-fluid materials.

The first source of unsteadiness was the insertion of a grain or grain fragment between the fixed upper plate and the rotating sidewall. This caused a temporary jamming of the apparatus, and a subsequent 'ringing' type of behaviour after the grain was fractured or dislodged. If this behaviour was observed, the experiment was stopped, the grain fragment removed (if it could be found), and the experiment repeated.

The second observation of unsteadiness resembled a 'stick–slip' behaviour, and occurred at low shear rates. When this behaviour occurred, the rotation rate alternately decreased and increased. This type of behaviour has been reported by Cheng & Richmond (1978) for suspensions of grains, but it is not completely understood. In the present experiments, it is believed this behaviour results when the material alternately dilates and compresses, causing a decrease or increase in the material's resistance to shear. This behaviour was observed for glass spheres only at low shear rates near the threshold stress for initial yield. For sand, however, the behaviour was observed from the yield point up to shear rates of approximately 40 s^{-1}. Presumably this is related to the angularity of the sand, and the preferred orientation, or fabric, which the material develops during shearing. This interesting phenomenon merits future study, but, since it complicates the interpretation of the steady-flow problem, the shear rates were restricted to avoid this unsteady complication.

Cheng (1984) discusses the numerous problems associated with rheological studies of dense suspensions. Several complications may arise because of the geometrical constraints of a particular apparatus. Physical and economic considerations unfortunately preclude the construction of an apparatus in which the number of grains being studied approaches Avogadro's number. In the present experiments, for example, the width or depth of the annular trough ranges between approximately 10 and 40 grain diameters. Aside from the implications to a continuum-based theory, this adds a complication in interpreting and comparing the experimental results.

Because the apparatus dimensions were only one to two orders of magnitude greater than a grain diameter, the grains could not be packed as tightly as in an infinitely wide 'container'. The maximum obtainable volume concentration was 0.55 for the 1.85 mm spheres, 0.64 for the 1.1 mm spheres and 0.61 for the beach sand. At these concentrations the material could not be sheared without fracture of individual grains. As will be discussed later, the stresses generated by the shearing of a granular-fluid material are relatively insensitive to concentration below approximately 0.5, but increase rapidly as particle rubbing and interlocking become significant at concentrations above approximately 0.5. The precise value of the concentration at which these effects become important depends only upon the grain characteristics for an infinite expanse of grains. In a finite-sized apparatus, however, because the maximum possible concentration is limited by the apparatus, the range of concentration in which the stresses increase rapidly must be considered to also depend upon the apparatus. It is thus unclear how to compare data obtained in differing apparatus, or even to compare different-sized materials studied in the same apparatus.

Given this caveat, in a later section different data sets will be compared on the basis of a scaled concentration

$$\hat{N} = N N_{\infty} / N_{\mathrm{m}}, \tag{3b}$$

In air	Number of experimental runs	Range in N
1.1 mm fully shearing	36	0.37–0.56
1.1 mm partially shearing	50	0.29–0.61
1.85 mm fully shearing	18	0.44–0.49
1.85 mm partially shearing	20	0.30–0.53
0.55 mm sand partially shearing	22	0.43–0.58
Subtotal	46	
In water		
1.1 mm fully shearing*	18	0.55–0.58
1.1 mm partially shearing	10	0.64–0.67
1.85 mm fully shearing	12	0.50–0.51
1.85 mm partially shearing	17	0.50–0.53
0.55 mm sand partially shearing*	19	0.56–0.59
Subtotal	76	
Total	222	

TABLE 4. Summary of the number of experimental runs of each type, and the range in volume concentrations in the experiments. * indicates lower-quality experiments.

where N_m is the maximum measured concentration of a given material in a given apparatus, and N_∞ is the asymptotic limit of N_m as the container dimensions approach infinity. For monosized spheres N_∞ is approximately 0.65 (Brown & Richards 1970).

The scaled concentration also provides a method to account for the errors implicit in measuring the volume concentration when the precise location of a boundary is inadequately defined. For example, if a boundary is roughened by irregularly spaced roughness elements, it is unclear what location defines the 'boundary' of the flow. Any errors introduced by using a certain level for the location of the boundary will be manifested in measurements of both N_m and N. By adopting the use of the scaled concentration, this error is compensated for.

When different data sets are compared using the suggested scaled concentration, it should be realized that this scaling is only an approximate attempt to account for the effects of a finite-sized apparatus. Because the stresses are strongly dependent upon the concentration above approximately 0.5, even small errors in the measurement of N can result in widely differing values of the stress.

2.6. *Quantity and quality of data*

A summary of the quantity of the various experiments is given in table 4. Most of the experiments involving glass spheres were consistent and repeatable. These data reveal clear trends and form the basis of most of the discussion in §3.

In contrast, the experiments on sand, and those on 1.1 mm glass spheres with water, were less consistent. The shear cell was designed to study materials having diameters ranging from 0.5 to 3 mm. Natural beach sand, being somewhat angular, has some cross-sections much less than 0.5 mm. In order to prevent the sharp edges of the sand grains from jamming into the gap between the Delrin lip and the sidewall, the Delrin was replaced with a tighter-fitting and more durable stainless-steel lip. This worked satisfactorily for dry sand, but caused a secondary problem for the wet experiments. Normally when shearing commences the granular-fluid material dilates, forcing the upper plate to move vertically. Air or water flows downward between the lip of the upper plate and the sidewall to fill the voids created between the grains. For the experiments on wet sand, water could not flow freely between the stainless-steel lip and the replaceable polycarbonate lining on the inside of the outer wall. The result was that the lining deformed, creating a tight seal with the steel lip. A pressure deficit was created within the annular trough, which prevented dilatation. This introduced a large uncertainty in the value of the normal stress, and lesser uncertainties for the shear stress and volume concentration. This set of data (sand–water) must be considered to be of poor quality.

Experiments on sand in air had similar problems related to the frictional nature of angular quartz grains. The interaction with the sidewall was frictional, so the 'slippery' boundary condition at the sidewall is suspect. The measurements of the shear stress could have been biased by this effect, since stress could be transferred from the sidewalls to the granular material.

The other set of experiments in which the quality is suspect involves the full shearing of 1.1 mm glass spheres in water. In order to shear the material fully the minimum depth of the annular gap had to be decreased by raising the lower boundary. This was done by inserting into the annulus a 2 cm thick Plexiglas toroid. In retrospect it appears that this toroid might have slipped during the experiments, thus introducing uncertainty to the shear-rate measurements. A second problem for this set of experiments is that significant secondary flows were probably set up in the interstitial fluid. Although the centrifugal forces were small compared with the intergranular forces, they were large compared with the interstitial fluid pressure. In response to the centrifugal force, the fluid should flow outward to develop a radial pressure gradient. This could have provided some buoyancy to the upper plate, and consequently some error in the normal-stress calculation. Furthermore, it is possible an air gap existed in the shearing region. For these reasons, this data set should be regarded with some suspicion.

In summary, the 124 runs involving glass spheres in air, the 29 runs involving 1.85 mm glass spheres in water, and the 10 partially shearing runs involving 1.1 mm glass spheres in water are thought to be of good quality. The 41 runs involving sand and the 18 fully shearing runs involving 1.1 mm glass spheres in water are of lower quality.

3. Review of theory

3.1. *Constitutive behaviour of shear flows*

Bagnold (1941, 1954) was first to demonstrate the importance of granular collisions to explain the non-Newtonian nature of rapid granular-fluid shear flows. By considering the mean paths of grains undergoing rapid shear deformation, Bagnold recognized that both the momentum transferred per collision and the frequency of granular collisions are proportional to the mean shear rate, resulting in tangential and normal stresses quadratic in the mean shear rate. Bagnold experimentally verified this quadratic dependence of the stresses, as well as the dependence of the stresses upon the volume concentration, by studying the flow of neutrally buoyant wax spheres sheared between concentric cylinders. Combining theory with observations, Bagnold proposed the following relations:

$$T_{xz} \propto \rho_s\left(\lambda D \frac{dU}{dz}\right)^2, \quad T_{zz} \propto \rho_s\left(\lambda D \frac{dU}{dz}\right)^2. \tag{4a, b}$$

These relations were experimentally verified for $B \geqslant 450$ and $0.14 \leqslant N \leqslant 0.60$ (or $1.4 \leqslant \lambda \leqslant 14$), where B is defined by (2). The stress ratio T_{xz}/T_{zz} ($\equiv \tan\phi$) was found to be equal to about 0.32, or tan 18°. Bagnold suggested that this ratio is constant in the grain-inertial regime, where grain-to-grain forces dominate the dynamics.

Bagnold's results have stood for nearly 30 years for the conditions under which they were formulated: steady, uniform, simple shear flow of neutrally buoyant granular materials. However, in several situations involving more complex flows or boundary conditions, the application of Bagnold's relations leads to unrealistic constraints. Complications arise from one of two sources. First, the stress ratio found to be constant in Bagnold's grain-inertia regime using neutrally buoyant particles conflicts with later observations of flows of non-neutrally buoyant particles in which the stress ratio varies. Secondly, and more importantly, the stresses vanish for vanishing mean-velocity gradient because in Bagnold's formulation there is no source for granular velocity fluctuations other than the mean shear.

Savage & Jeffrey (1981), and Jenkins & Savage (1983) derive constitutive relations for the rapid shear flow of cohesionless spheres by explicitly considering the fluctuating component of velocity in calculating the momentum transfer due to binary granular collisions. Following the examples of Ogawa (1978, 1980) they consider the fluctuating component of the velocity to be a mechanical, or 'macroscopic' temperature, hereinafter referred to as θ, the pseudotemperature. The development is analogous to the kinetic theory of gases, with the primary differences being that in a granular fluid the collisions are driven by the inhomogeneity in the mean flow, and energy is lost via internal dissipation during imperfectly elastic collisions. One of the essential ingredients in the development of these theories was the recognition that the distribution of collisions is anisotropic in a shear flow even though the distribution of grains may be homogeneous. This is a consequence of the fact that both the mean motion and the fluctuating motions are occurring at approximately the same spatial scales.

Savage, Jenkins and Jeffrey's approach is to define the stresses and energy dissipation in terms of integrals over all probable collisions. These integrations can be carried out by assuming a single-particle velocity distribution function, and a complete pair distribution function for two particles at contact. Savage & Jeffrey assume the single-particle velocity distribution function to be Maxwellian about the mean velocity. They derive a complete pair-distribution function to be a function of $R = D(\overline{v'^2})^{-\frac{1}{2}}\,\mathrm{d}U/\mathrm{d}z$, the ratio of the mean shear characteristic velocity to the r.m.s. precollisional velocity fluctuation. They then numerically integrate the collision integrals for arbitrary R. Jenkins & Savage modified the theory by assuming the collision distribution function A to be linear in the velocity gradient. They suggest using the form

$$A = 1-\alpha \boldsymbol{K}\cdot \boldsymbol{U}_{12}(\pi\theta)^{-\frac{1}{2}}, \tag{5}$$

where $\boldsymbol{U}_{12}$ is the relative velocity of colliding grains, $\boldsymbol{K}$ is the unit vector connecting the centre of the two grains at collision, and $\alpha(N)$ is an unknown function. This is a simple form which captures the ideas about collision anisotropy expressed above. It is equivalent to Savage & Jeffrey's formulation in the limit of small R and $\alpha = 1$.

For shear flows, Jenkins & Savage derive the following expressions for the stresses and energy balance:

$$T_{xz} = -0.2\kappa(2+\alpha)\frac{\mathrm{d}U}{\mathrm{d}z}, \quad T_{zz} = \kappa(\pi\theta)^{\frac{1}{2}}D^{-1}, \quad \frac{T_{xz}}{T_{zz}} = (2+\alpha)\frac{\mathrm{d}U}{\mathrm{d}z}[5D(\pi\theta)^{\frac{1}{2}}]^{-1}, \qquad (6a\text{–}c)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)+T_{xz}\frac{\mathrm{d}U}{\mathrm{d}z}-6(1-e)\kappa\theta D^{-2} = 0, \tag{7}$$

where $$\kappa = 2\rho_s N^2 g_0 D(1+e)\,\theta^{\frac{1}{2}}\pi^{-\frac{1}{2}}, \tag{8}$$

$$g_0 = (1-N)^{-1} + 1.5N(1-N)^{-2} + 0.5N^2(1-N)^{-3} = 0.5(2-N)(1-N)^{-3}, \tag{9}$$

and e is the coefficient of elasticity of the granular material.

In (7) the first term represents the diffusion of mechanical heat, the second term is the generation of mechanical heat by the mean shear, and the last term is the dissipation into actual heat due to imperfectly elastic granular collisions. Somewhat analogous to the generation of turbulence in a fluid, there is a cascade of energy from the mean shear into fluctuating kinetic energy, and finally into true heat, as suggested by McTigue (1979) and Jenkins & Cowin (1979).

Jenkins & Savage give solutions to these equations corresponding to shear flow between parallel plates for the limiting case where self-weight of the grains is small compared with the applied normal stress, and the boundaries are passive ($\theta' = 0$). In these solutions the stresses are given by

$$T_{xz} = \rho_s D^2 F\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 (2+\alpha)^{\frac{3}{2}}(1+e)\,[750(1-e)\,\pi]^{-\frac{1}{2}}, \tag{10a}$$

$$T_{zz} = \rho_s D^2 F\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 (2+\alpha)(1+e)\,[30(1-e)]^{-1}, \tag{10b}$$

$$\frac{T_{xz}}{T_{zz}} = [6(2+\alpha)(1-e)]^{\frac{1}{2}}(5\pi)^{-\frac{1}{2}}, \tag{10c}$$

$$\theta = \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 (2+\alpha)\,[30(1-e)]^{-1}, \tag{10d}$$

where $$F = N^2(2-N)(1-N)^{-3}. \tag{11}$$

These results are consistent with Bagnold's with regard to the dependence of the stresses upon the grain density, diameter and the mean shear rate. The primary difference is that Bagnold's empirically determined dependence of the stresses upon the volume concentration and grain elasticity has been replaced by the analytical expressions of (10) and (11).

3.2. *Flow in a semi-infinite granular bed*

Consider a bed of cohesionless grains oriented with its surface perpendicular to the gravity field, as shown in figure 2. If traction $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_n$ is applied to the surface of the bed, the granular material will deform. Consider the case of steady shear deformation in which the mean paths of all moving grains are horizontal, and there are no gradients in the direction of mean flow. In this case the mass balance is identically satisfied, and the linear momentum equations reduce to

$$T_{xz} = \tau_0, \quad T_{zz}(z) = \tau_n + \int_0^z \rho_s N g\,\mathrm{d}z. \tag{12a, b}$$

The shear stress remains constant at all depths, equal to the applied shear stress. If the shear stress were to change with depth, the deformation would be unsteady. The normal stress is an increasing function of depth, owing to the self-weight of the grains. The ratio of the shear stress to the normal stress therefore decreases with increasing depth in the bed. Following the well-established Coulomb yield criterion for static yield (Coulomb 1773), it is useful to define a Coulomb yield criterion for dynamic yield, corresponding to the limiting value of the stress ratio at which motion ceases,

$$T_{xz}/T_{zz} = \tan\phi_r. \tag{13}$$

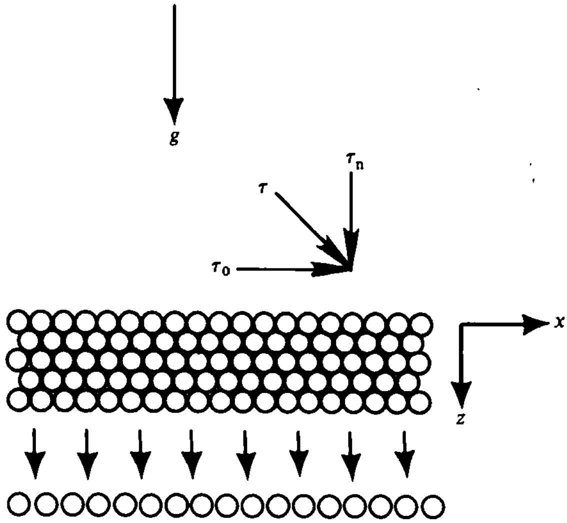

FIGURE 2. Definition sketch for a semi-infinite granular bed subject to surface traction and gravity forces.

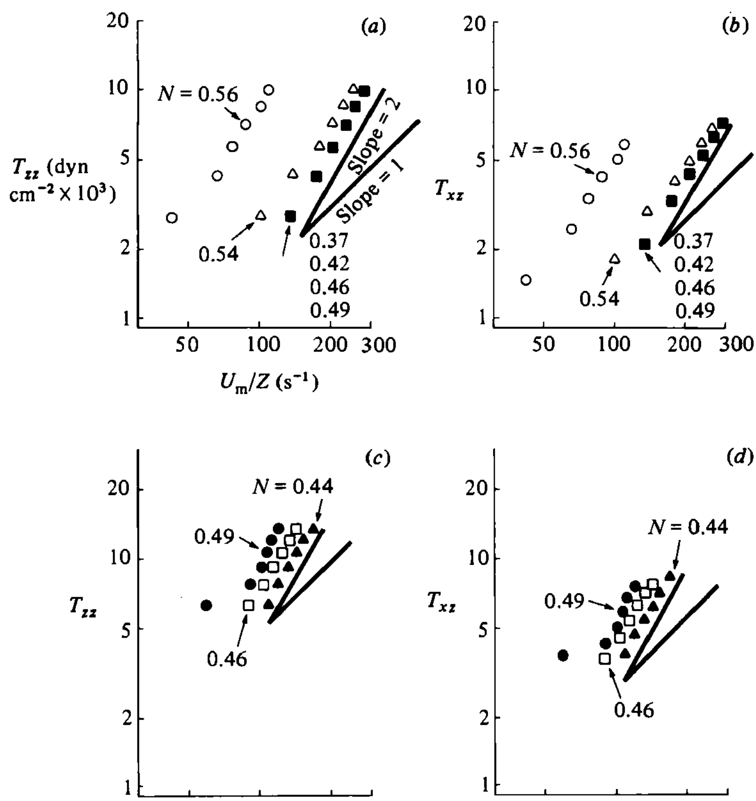

FIGURE 3. Shear and normal stresses as a function of shear rate for glass spheres in air: (a), (b) 1.1 mm; (c), (d) 1.85 mm. A slope of 2 (at constant N) indicates a quadratic dependence of the stresses upon the shear rate.

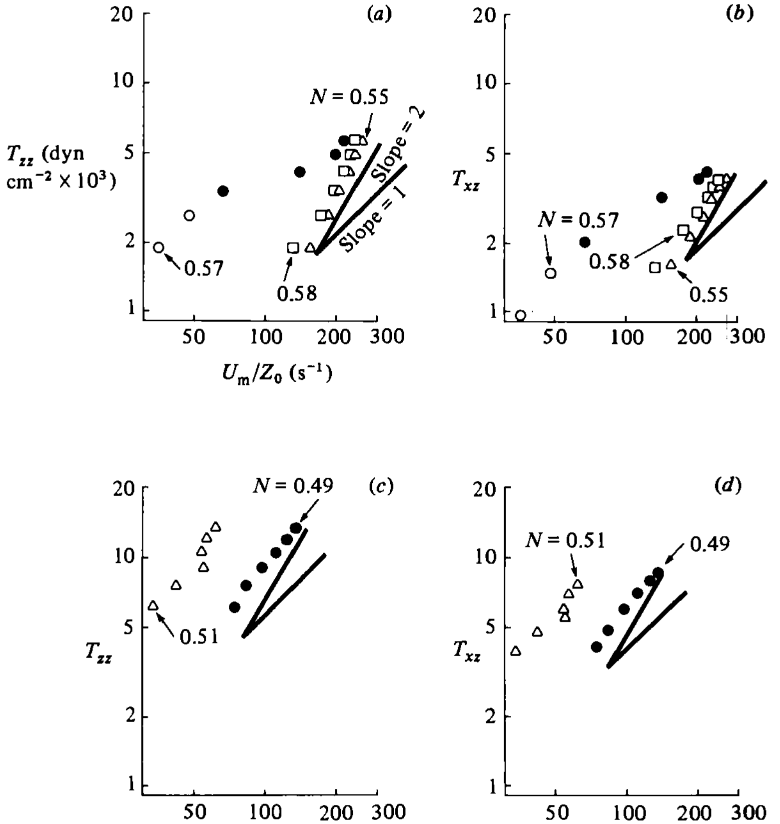

FIGURE 4. Shear and normal stresses as a function of shear rate for glass spheres in water: (*a*), (*b*) 1.1 mm; (*c*), (*d*) 1.85 mm. Open circles indicate data lie in Bagnold's transition region ($B < 450$). Slopes (at constant N) generally lie between 1 and 2.

4. Observations and discussion

The experimental results have implications for several aspects of granular-fluid flow including the constitutive behaviour, the existence of a finite thickness of motion, and some effects of the differing interstitial fluids.

4.1. *Constitutive behaviour*

The influence of the shear rate and the volume concentration upon the stresses can be separately evaluated by examining the fully shearing experiments. In these experiments the shear rate and the volume concentration were varied independently.

The shear stress and the normal stress are shown as a function of the nominal shear rate U_m/Z_0, and the nominal volume concentration N in figures 3 and 4. As expected, the stresses are increasing functions of both the volume concentration and the shear rate. According to the collision theories discussed in §3, the stresses should have a quadratic dependence upon the shear rate. In the experiments for which air is the interstitial fluid, the slopes (for constant N) are approximately 2, supporting the predicted quadratic relationship. For the experiments in which water is the interstitial fluid, the slopes vary between 1 and 2. This is consistent with earlier calculations

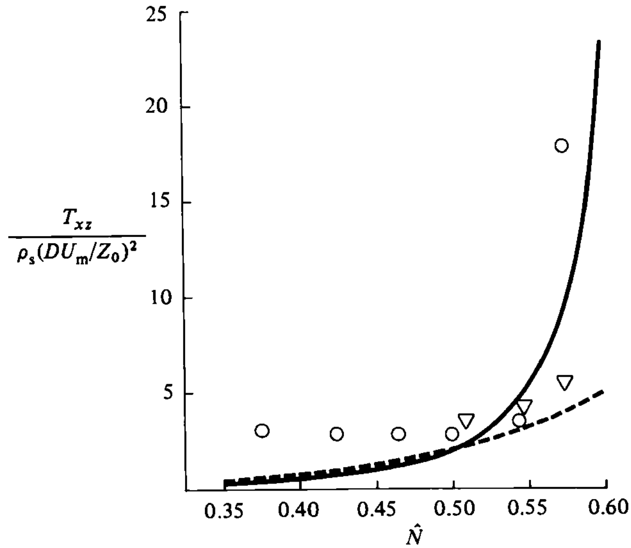

FIGURE 5. Non-dimensional shear stress as a function of scaled volume concentration for fully shearing glass spheres in air: ○, 1.1 mm; ▽, 1.85 mm. The solid curve represents an empirical fit to the relations of Bagnold (1954), and the broken curve represents the theory of Jenkins & Savage (1983).

showing that some of these experiments lie in Bagnold's transition regime, where the interstitial fluid may affect the overall dynamics. The open circles in figure 4 indicate that the data are in the transition region, using the values of B obtained by Bagnold. In terms of the kinetic models discussed, it appears that the water partially dampens the grain trajectories, and hence the velocity fluctuations of the grains, thus decreasing the impact during collisions.

The dependence of the stresses upon the volume concentration for the fully shearing 'dry' experiments are shown in figures 5 and 6 as plotted points, together with the prediction curves of Bagnold and Jenkins & Savage. The stresses have been non-dimensionalized by $\rho_s D^2(U_m/Z_0)^2$, in accordance with collisional theory. The data in figures 5 and 6 are the same as those of figure 3, but in figures 5 and 6 each point represents several experimental runs at the same concentration and differing shear rates. The abscissa in these figures is the scaled concentration $\hat{N}$, in accordance with the previous discussion in §2.5. The dry partially shearing experiments reveal the same pattern for the nondimensional stresses as the fully shearing experiments. For example, the non-dimensional shear stresses for the partially shearing 1.85 mm beads are shown in figure 7, where each point represents a single experimental run.

The coefficient of restitution e used in Jenkins & Savage's prediction is 0.9, and $\alpha = 1$. Although the values of e and α are not confidently known, it is unlikely that they could be higher than the values used here. If α were higher, the theory becomes less valid because the anisotropy term in (5) becomes too large. For nearly elastic particles e could approach 1, but that would result in the predicted stress ratio (10c) being much smaller than those observed. The curves in figures 5 and 6 labelled 'Bagnold' represent an empirical fit of (4) to the present data, where N^* in (3) has been set equal to N_∞ (0.65), and the proportionality constants of (4) have been fitted by eye.

As seen in figures 5–7, the stresses are only weakly dependent on the concentration up to approximately $\hat{N} = 0.5$. The lack of dependence of the stresses for $\hat{N} < 0.5$ for

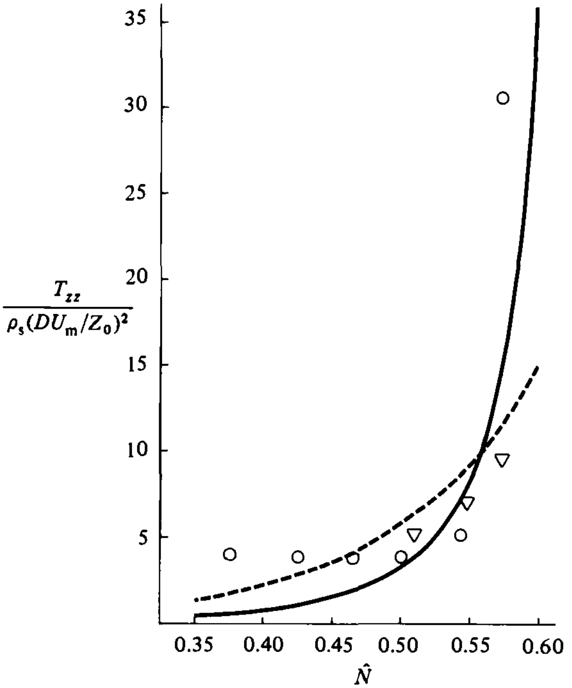

FIGURE 6. Non-dimensional normal stress (see caption for figure 5).

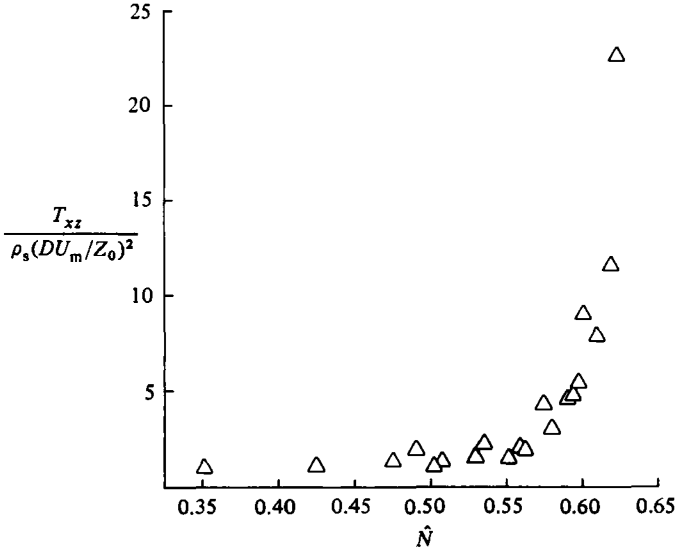

FIGURE 7. Non-dimensional shear stress as a function of scaled volume concentration for partially shearing 1.85 mm glass spheres in air.

the smaller spheres might have been due to an air gap in the annular trough. Savage & Sayed (1984) observed the same behaviour in their similar apparatus and offered this explanation. Above concentrations of approximately 0.5 the stresses increase rapidly. The Bagnold curves describe this behaviour quite well. The Jenkins & Savage curves predict the proper trend, but the increase in stress above $\hat{N} = 0.55$ is

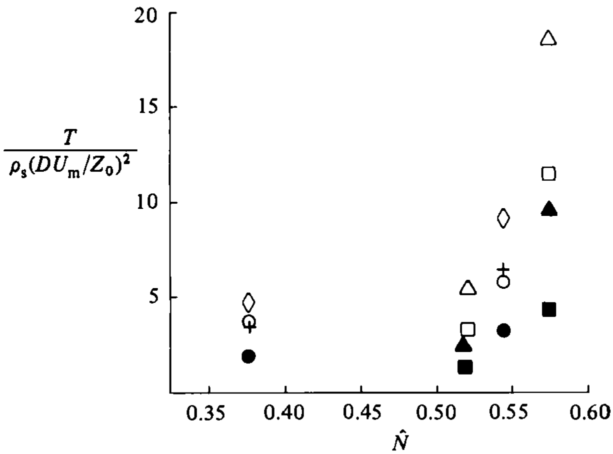

FIGURE 8. A comparison of the non-dimensional stresses for selected partially and fully shearing runs indicates higher stresses for the fully shearing runs. Symbols:

T_{zz}	T_{xz}	
△	□	1.85 mm fully shearing in air
▲	■	1.85 mm partially shearing in air
◇	○	1.1 mm fully shearing in air
+	●	1.1 mm partially shearing in air

not sharp enough. Note that the dependence of the stresses upon the concentration was empirically determined by Bagnold. In contrast, Jenkins & Savage derive the dependence on N in the context of their kinetic theory. It is not surprising that the kinetic theory underpredicts the stresses at high N, because multiple collisions and sliding friction between grains are expected to become significant.

It is interesting that at equivalent mean concentrations the non-dimensional stresses generated in the fully shearing experiments tended to be higher than those generated in the partially shearing experiments. This is illustrated in figure 8, which compares selected pairs of runs at equivalent mean concentrations. The differing stresses suggest that the boundaries may strongly influence the dynamics of the flow. Because grains at the lower boundary of the partially shearing experiments were movable, the boundary may have acted in a 'cooler' manner than a rigid boundary. Local slip between impacted grains could result in a loss of mechanical heat. At equivalent shear rates and concentrations, the stresses associated with movable grains at the lower boundary are therefore lower than those associated with fixed grain boundaries because the pseudotemperature of the entire flow is lower.

The 18 experiments on fully shearing flows of 1.85 mm glass spheres in air are quite similar to some of the experiments reported by Savage & Sayed (1984). The only significant difference between the experimental arrangements is that Savage & Sayed roughened the shearing surfaces by gluing sandpaper to a cardboard liner, which was in turn glued in place with rubber cement. In the present experiments the actual grains being studied were attached to the surfaces with epoxy. Because both sets of data indicate the quadratic relationship between shear rate and stress, it would appear that the different surface characteristics both results in collisional-type flows.

However, the actual stress levels were greater in the present experiments than they were in Savage & Sayed's experiments (at equivalent concentrations and shear rates). For example, the shear stress developed by shearing 1.85 mm glass spheres at a concentration $\hat{N}$ of approximately 0.58 were about 3 times higher in the present

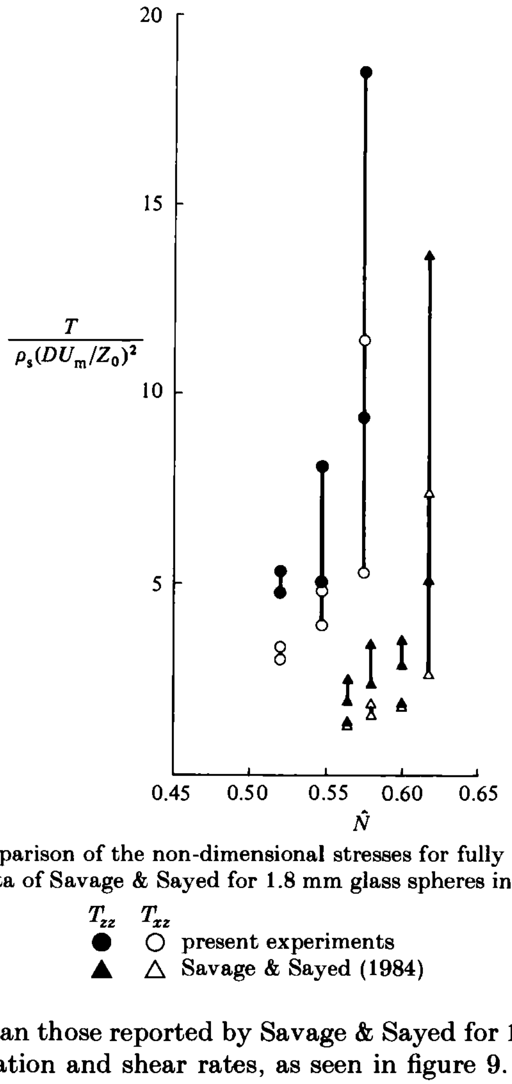

FIGURE 9. A comparison of the non-dimensional stresses for fully shearing 1.85 mm glass spheres in air and the data of Savage & Sayed for 1.8 mm glass spheres in air. Symbols:

T_{zz}	T_{xz}	
●	○	present experiments
▲	△	Savage & Sayed (1984)

experiments than those reported by Savage & Sayed for 1.8 mm glass spheres at the same concentration and shear rates, as seen in figure 9. This difference is believed to result from the difference in materials at the upper and lower boundaries of the flow. The interaction at a boundary presumably depends upon the frictional and elastic characteristics of the interacting materials. It is possible that a boundary may act as a direct source or sink of translational (or rotational) fluctuation energy. The difference in the stress levels measured by Savage & Sayed and those measured in the present experiments may be partially explained if the rigid grain boundaries in the present experiments directly provided more fluctuational energy than the sandpaper boundaries in the experiments of Savage & Sayed.

A second explanation for the difference in stresses is suggested by the observation that the sandpapered boundaries significantly scratched the glass beads (Savage, personal communication). The recent theory of Lun & Savage (1984) shows that because particles with rough surfaces tend to have more energy in rotational motion than particles with smooth surfaces, the shearing of rough particles generates lower stresses than a similar shearing of smooth particles. However, for particles with a coefficient of restitution $e = 0.9$, the theorized reduction in the normal stress (65%) due to rough particles is much greater than the reduction in shear stress (35%). This

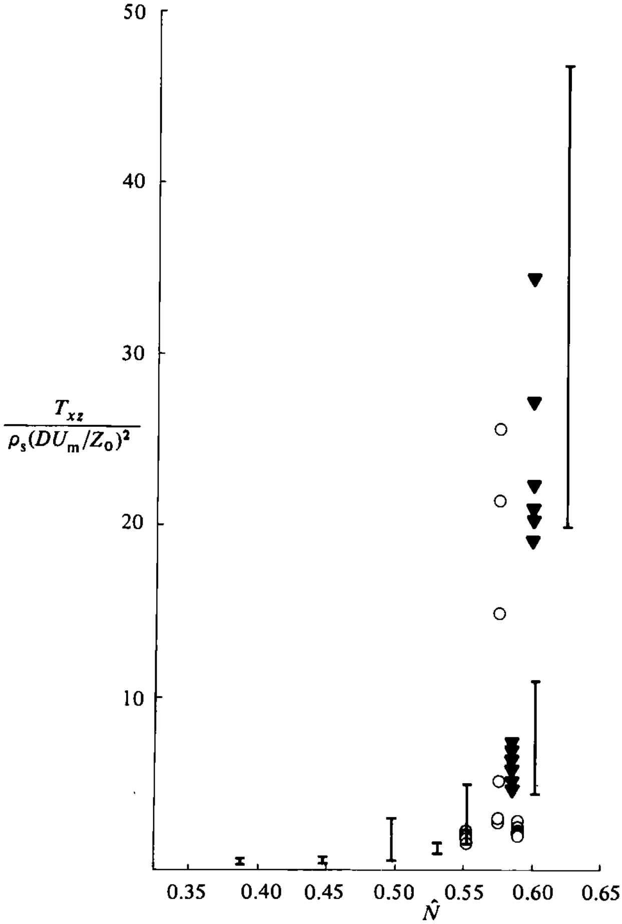

FIGURE 10. A comparison of the non-dimensional shear stress for fully shearing glass spheres in water with the data of Bagnold (1954) for 1.32 mm wax spheres in water. ——, Bagnold's data; ○, 1.1 mm; ▼, 1.85 mm, present experiments.

explanation alone cannot therefore explain the magnitude of the observed differences in both the normal and shear stresses. Probably both the decreased pseudotemperature and the increased surface roughness contribute to the lower stresses.

The present experiments in which water is the interstitial fluid can be compared to Bagnold's (1954) measurements, although it should be noted that both the appratus and the materials differ. Figure 10 compares the non-dimensional shear stresses as a function of $\hat{N}$. For Bagnold's data N_m is assumed to be 0.65, so $\hat{N} = N$. The data for the larger spheres tend to agree in both trend and magnitude with Bagnold's data. This agreement is quite encouraging, although considering the experimental differences it may be somewhat fortuitous. The inconsistencies in the data obtained with smaller spheres could result from either errors in measuring the volume concentration, or from the experimental problems discussed in §2.6, and thus should be regarded skeptically.

	Z_0/D		Immersed weight (dyn cm^{-2})	
	Mean	Standard deviation	Mean	Standard deviation
1.1 mm glass spheres in air (50)	9.0	1.8	1411	382
1.1 mm glass spheres in water (10)	14.2	2.0	1470	196
1.85 mm glass spheres in air (20)	7.1	1.2	1646	402
1.85 mm glass spheres in water (17)	12.7	1.5	2127	235
0.55 mm sand in air (22)	6.9	0.9	500	78
	10.4	2.2	529	108

TABLE 5. Non-dimensional thickness of the shearing region and the immersed weight of the shearing grains

4.2. *The thickness of the shearing layer*

A finite thickness of the shearing layer was observed (over a large range in applied stress and shear rate) for all materials studied. The non-dimensional thickness and the immersed weights (per unit area) of the moving grains are given in table 5.

The thickness of motion was generally between 5 and 10 grain diameters for flows in air, and between 10 and 15 grain diameters for flows in water. The larger thicknesses observed for flows in water support the hypothesis that the thickness of motion is limited by the self-weight of the grains, since the water provides buoyancy to reduce the weight of a grain. The ratio of the mean thickness in water to that in air for a given material was found to be approximately equal to the ratio of a grain's weight in air to its weight in water.

The variation in the thickness of motion for each material was small compared to the variations in the applied stress and shear rate. For example, for the 50 runs on 1.1 mm glass spheres in air, the standard deviation of the shear rate or applied normal stress was approximately 40 % of its mean, while the standard deviation in the thickness of motion was only 19 % of its mean.

The observations of a finite thickness of motion has severe implications regarding the application of continuum-based flow theories to rapid flows that are acted upon by gravity, because it appears that flows of this nature are limited to thicknesses of only a few grain diameters. It is quite likely that in the present experiments (and in natural flows) there exists a 'quasi-static' transition region, in which grains may slide and roll in close contact with one another, between the rigid region and the collisional region. In order to develop a complete description of such flows, these three regions must be linked together in a rational, physically reasonable, way.

4.3. *The stress ratio*

The stress ratio T_{xz}/T_{zz}, evaluated at the lower boundary of the flow, reveals several consistent trends. The mean stress ratios, and their standard deviations, are given in table 6. Using values of $e = 0.9$ and $\alpha = 1$ in (10*c*), the theory of Jenkins & Savage predicts a stress ratio of 0.34. This value is lower than the measurements, reflecting the underprediction of the shear stress (see figure 5).

The standard deviations of the stress ratios are approximately 10 % of the mean, supporting the concept of a nearly constant stress ratio (over a range of shear rates and applied normal stresses) for a given material. The correlation of the stress ratio

Stress ratios (mean (standard deviation))	In air	In water
1.1 mm spheres		
fully shearing	0.628 (9.9 %)	0.640 (13.1 %)
partially shearing	0.437 (12.5 %)	0.436 (17.8 %)
1.85 mm spheres		
fully shearing	0.533 (3.9 %)	0.590 (5.9 %)
partially shearing	0.409 (8.5 %)	0.528 (5.3 %)
0.55 mm sand		
partially shearing	0.680 (13.7 %)	0.891 (15.8 %)

TABLE 6. The measured stress ratios

with the shear rate and volume concentration is discussed in detail in Hanes & Inman (1984).

Perhaps the most noticeable result is that the stress ratios for the fully shearing experiments are significantly higher than those resulting from the partially shearing flows. In terms of the momentum arguments presented earlier, this observation indicates that, if there were erodible grains at the lower boundary of the fully shearing flow, they would be mobilized into the shear flow. The solid lower boundary, although fully stressed from above, cannot partake in the granular fluctuations which would result in a higher normal stress. Thus there is a deficiency in the normal stress, and the stress ratios for the fully shearing flows are higher than the ratios for partially shearing flows.

This observation may explain why Savage & Sayed's (1984) measurements of the stress ratio were higher than the dynamic angle of repose of the material. Because all of their experiments involved fully shearing conditions, the values of the stress ratio were higher than expected for a stress state close to the yield point. The dynamic friction angles measured in the present experiments, expressed as the inverse tangent of the stress ratio, are compared to the dynamic angle of repose for the various materials used in the present experiments in table 7. The angle of repose generally lies between the friction angle for partially shearing flow and the friction angle for fully shearing flow.

In terms of a kinetic theory, the differing stress ratios suggest that the angular distribution of collisions is different in the partially and fully shearing flows. Hanes & Inman (1984) discuss this point in detail, and suggest that the differences in the nature of the lower boundaries of the fully and partially shearing flows alters the collisional distribution and hence the stress ratio throughout the entire depth of the flow.

An interesting result in the measurements of the stress ratio is that the ratios for experiments on 1.85 mm glass spheres in which water was the interstitial fluid are higher than those measured for experiments in which air was the interstitial fluid. There are at least two explanations for this observation. The simplest explanation is that the deformation of the interstitial water itself generates a much higher shear stress than that generated by the deformation of air. This stress adds to the shear stress generated by granular interactions, resulting in a higher stress ratio. However, because the fluid-generated stresses are much smaller than the granular stresses, this explanation is inadequate. A second explanation is offered in the context of the

	Mean measured dynamic friction angles		Dynamic angle of repose
	In air	In water	In air
1.1 mm spheres			
fully shearing	32°	33°	26°
partially shearing	24°	24°	—
1.85 mm spheres			
fully shearing	28°	31°	28°
partially shearing	22°	28°	—

TABLE 7. The measured friction angles and the dynamic angle of repose

kinetic theories discussed previously. It follows from (6)–(9) that the normal stress varies directly with the pseudotemperature, while the shear stress depends upon the product of the shear rate and the square root of the pseudotemperature. One of the effects water has on the flow is that it decreases the pseudotemperature by dampening vibrational motions, while maintaining the same mean velocities. Thus the stress ratio, which varies as the inverse of the square root of the pseudotemperature, is higher when water is the interstitial fluid. Probably both the additional shear stresses and the dampening of granular vibrations due to the interstitial-fluid viscosity contribute to the higher stress ratios measured when water was the interstitial fluid. It should be noted that the same trend was not observed for the smaller glass spheres.

5. Conclusions

Observations of the shear flow of granular-fluid materials demonstrate that at sufficiently high shear rates the stresses are quadratically dependent upon the mean shear rate (at a constant volume concentration). These measurements confirm many of the observations of Savage & Sayed (1984), which in turn supports Bagnold's hypothesis that granular collisions comprise the primary mechanism by which momentum is transferred in these flows.

The stresses were found (in the present study) to be weakly dependent on the volume concentration up to approximately 0.5. Above this concentration, the stresses were strongly dependent on the volume concentration. Sometimes the stresses increased by an order of magnitude with less than a 10% increase in the concentration.

These observations are in general agreement with the relations of Bagnold (1954) and the kinetic theory of Jenkins & Savage (1983). The kinetic theory predicts an increase in the stress levels with increasing concentration, but the predicted increase is smaller than that observed for volume concentrations greater than about 0.55. The shear stress and thus the stress ratio tends to be underpredicted by the kinetic theory. The kinetic approach will undoubtedly be improved by the inclusion of such phenomena as particle spin, friction and multiple collisions.

The characteristics of the boundaries appear to have a dramatic influence upon the nature of the flow. The rigid boundaries of the fully shearing experiments result in both higher stresses and a higher stress ratio than the movable boundaries of the partially shearing experiments. In the context of a kinetic theory, these observations are consistent if the rigid boundaries result in a higher pseudotemperature and a more oblique angular distribution of collisions. It is suggested that the present fully shearing data indicate higher stresses than the data of Savage & Sayed (1984) because the rigid glass-sphere boundaries resulted in a higher pseudotemperature than the sandpaper boundaries of Savage & Sayed. The influence of the boundary characteristics on the dynamics of rapidly flowing granular-fluid materials is clearly an area which requires further study.

The experiments clearly demonstrate the existence of an internal boundary separating a shearing region from a rigid region. The thickness of the shearing region was between five and fifteen grain diameters. This phenomenon appears to occur as a simple result of momentum conservation in a gravity field, and a Coulomb-type yield criterion. The stresses at the internal boundary were found to conform to a dynamic Coulomb yield criterion to within approximately 10%.

The finite thickness of the shearing layer illustrates an important problem: the rapid-flow regime needs to be systematically related to the less intense quasi-static, or plastic, regime of deformation in a continuous and physically meaningful way. Most geophysical flows, and many industrial flows, involve a transition between the two regimes. The limited thicknesses also bring into question the appropriateness of treating the material as a continuum, because the entire flow is limited to approximately five to ten grain layers.

Measurements of the stress ratios and of the constitutive behaviour indicate that the interstitial fluid probably has a significant influence in many flows of interest. This is evidenced by the relatively higher stress ratios and the more-linear dependence of the stresses on the shear rate for experiments in which water was the interstitial fluid. Probably both the stresses generated by the deformation of the water, and the dampening of grain trajectories by the water, have a significant influence upon the mechanics of the mixture, as suggested by Bagnold. Although there have been some attempts to explain such flows (e.g. Ackerman & Shen 1982), more rigorous theoretical bases for flows of this nature, where both the granular collisions and the interstitial-fluid effects are important, need to be developed. The limiting cases of a suspension of grains on one side, and the rapid collisional flow on the other, are fairly well formulated, and provide the asymptotic limits for future theories.

D. M. Hanes would like to thank J. T. Jenkins, S. B. Savage and R. J. Seymour for helpful discussions. This manuscript was written while D. M. Hanes was a Killam Postdoctoral Fellow at Dahlhousie University. Financial support was provided by the U.S. Office of Naval Research, the National Oceanic and Atmospheric Administration, and the Killam Trust.

REFERENCES

Ackerman, N. L. & Shen, H. 1982 Stresses in rapidly sheared fluid–solid mixtures. *J. Engng Mech. Div. ASCE* **108**, 95–113.

Bagnold, R. A. 1941 *The Physics of Blown Sand and Desert Dunes*. Chapman & Hall.

Bagnold, R. A. 1954 Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear, *Proc. R. Soc. Lond.* A **225**, 49–63.

Bagnold, R. A. 1965 The flow of cohesionless grains in fluids, *Phil. Trans. R. Soc. Lond.* A **249**, 235–297.

Bagnold, R. A. 1966 An approach to the sediment transport problem from general physics. *US Geol. Survey Prof. Paper* 422-J.

Bailard, J. A. 1978 An experimental study of granular-fluid flow. Ph.D. dissertation, University of California at San Diego.

Bailard, J. A. & Inman, D. L. 1979 A re-examination of Bagnold's granular fluid model and bed load transport equation. *J. Geophys. Res.* **84**, 7827–7833.

Brown, R. L. & Richards, J. C. 1970 *Principles of Powder Mechanics*. Pergamon.

Campbell, C. S. 1982 Shear flows of granular materials. Ph.D. dissertation, California Institute of Technology.

Campbell, C. S. & Brennen, C. E. 1982 Computer simulation of shear flows of granular materials. In *Proc. US–Japan Seminar on New Models and Constitutive Relations in the Mechanics of Granular Materials* (ed. J. T. Jenkins & M. Satake). Elsevier.

Carr, J. F. & Walker, D. M. 1968 An annular shear cell for granular materials. *Powder Tech.* **1**, 369–373.

Cheng, D. C. 1984 Further observations on the rheological behaviour of dense suspensions. *Powder Tech.* **37**, 255–273.

Cheng, D. C. & Richmond, R. A. 1978 Some observations on the rheological behavior of dense suspensions. *Rheol. Acta* **17**, 447–453.

Coulomb, C. A. 1773 Essai sur une application des règles de maximis et minimis à quelques problèmes de statique relatifs à l'architecture. *Acad. R. Sci. Mém. Math. Phys. par Divers Savants* **7**, 343–382.

Goodman, M. A. & Cowin, S. C. 1972 A continuum theory for granular material. *Arch. Rat. Mech. Anal.* **44**, 239–266.

Hanes, D. M. & Inman, D. L. 1984 A dynamic yield criterion for granular-fluid flows. *J. Geophys. Res.* (Submitted).

Inman, D. L. 1953 Areal and seasonal variations in beach and nearshore sediments at La Jolla, California. *Beach Erosion Board, Corps of Engrs Tech. Memo* 39.

Inman, D. L. & Hanes, D. M. 1980 Field measurements of bed and suspended load motion in the surf zone. In *Proc. 17th Intl Conf. on Coastal Engineering; abstracts vol.*

Jenkins, J. T. & Cowin, S. C. 1979 Theories for flowing granular materials. In *Mechanics Applied to the Transport of Bulk Materials. ASME* AMD-31, pp. 79–89.

Jenkins, J. T. & Satake, M. 1983 *Mechanics of Granular Materials: New Models and Constitutive Relations*. Elsevier.

Jenkins, J. T. & Savage, S. B. 1983 A theory for the rapid flow of identical, smooth, nearly elastic spherical particles. *J. Fluid Mech.* **130**, 197–202.

Komar, P. D. & Inman, D. L. 1970 Longshore sand transport on beaches. *J. Geophys. Res.* **76**, 5914–5927.

Lowe, D. R. 1976 Grain flow and grain flow deposits. *J. Sedim. Petrol.* **46**, 188–199.

Lun, C. K. K. & Savage, S. B. 1984 A simple kinetic theory for granular flow of rough, inelastic spherical particles. *Trans. ASME* E: *J. Appl. Mech.* (submitted).

Lun, C. K. K., Savage, S. B., Jeffrey, D. J. & Chepurniy, N. 1984 Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flow field. *J. Fluid Mech.* **140**, 223–256.

McTigue, D. F. 1982 A nonlinear constitutive model for granular materials: application to gravity flow. *Trans. ASME* E: *Appl. Mech.* **49**, 291–296.

Ogawa, S. 1978 Multitemperature theory of granular materials. In *Proc. US–Japan Seminar on Continuum-Mechanical and Statistical Approaches in the Mechanics of Granular Materials* (ed. S. C. Cowin & M. Satake, M.) pp. 208–217. Gakujutsu Bunken Fukyukai.

Ogawa, S., Umemura, A. & Oshima, N. 1983 On the equations of fully fluidized granular materials. *Z. angew. Math. Phys.* **31**, 483–493.

Owen, P. R. 1964 Saltation of uniform grains in air. *J. Fluid Mech.* **20**, 225–242.

Reynolds, O. 1885 On the dilatancy of media composed of rigid particles in contact. *Phil. Mag.* (5) **20**, 469–481.

Savage, S. B. 1978 Experiments on shear flows of cohesionless granular materials. In *Proc. US–Japan Seminar on Continuum-Mechanical and Statistical Approaches in the Mechanics of Granular Materials* (ed. S. C. Cowin & M. Satake, pp. 241–254. Gakujutsu Bunken Fukyukai.

SAVAGE, S. B. 1979 Gravity flow of cohesionless granular materials in chutes and channels. *J. Fluid Mech.* **92**, 53–96.

SAVAGE, S. C. & JEFFREY, D. J. 1981 The stress tensor in a granular flow at high shear rates. *J. Fluid Mech.* **110**, 255–272.

SAVAGE, S. B. & MCKEOWN, S. 1983 Shear stresses developed during rapid shear of concentrated suspensions of large spherical particles between concentric cylinders. *J. Fluid Mech.* **127**, 453–472.

SAVAGE, S. B. & SAYED, M. 1984 Stresses developed by dry cohesionless granular materials sheared in an annular shear cell. *J. Fluid Mech.* **142**, 391–430.

SAYED, M. 1980 Theoretical and experimental studies on dry cohesionless granular materials. Ph.D. dissertation, McGill University.

VERMEER, P. A. & LUGER, H. J. (eds) 1982 *Deformation and Failure of Granular Materials.* Balkema.